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On the intersection of the $\mathfrak{F}$-maximal subgroups and the generalized $\mathfrak{F}$-hypercentre of a finite group

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ABSTRACT

Let $\mathfrak{F}$ be a class of groups. A chief factor H/K of a group G is called *$\mathfrak{F}$-central in G* provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. We write $Z_{\pi\mathfrak{F}}(G)$ to denote the product of all normal subgroups of G whose G-chief factors of order divisible by at least one prime in π are $\mathfrak{F}$-central. We call $Z_{\pi\mathfrak{F}}(G)$ the $\pi\mathfrak{F}$-hypercentre of G. A subgroup U of a group G is called *$\mathfrak{F}$-maximal* in G provided that (a) $U \in \mathfrak{F}$, and (b) if $U \leqslant V \leqslant G$ and $V \in \mathfrak{F}$, then $U = V$. In this paper we study the properties of the intersection of all $\mathfrak{F}$-maximal subgroups of a finite group. In particular, we analyze the condition under which $Z_{\pi\mathfrak{F}}(G)$ coincides with the intersection of all $\mathfrak{F}$-maximal subgroups of G.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover p is always supposed to be a prime and π is a non-empty subset of the set $\mathbb{P}$ of all primes. We use $\mathfrak{G}_\pi$ ($\mathfrak{S}_\pi$) to denote the class of all π-groups (of all soluble π-groups, respectively). In particular, $\mathfrak{G}_p$ denotes the class of all p-groups; and we put that $\mathfrak{G}_\emptyset = \mathfrak{S}_\emptyset = (1)$ is the class of all identity groups. We

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also use $\mathcal{N}$, $\mathcal{U}$ and $\mathcal{S}$ to denote the classes of all nilpotent groups, of all supersoluble groups and of all soluble groups, respectively.

Let $\mathfrak{F}$ be a class of groups. A group G is said to be $\mathfrak{F}$-*critical* if G is not in $\mathfrak{F}$ but all proper subgroups of G are in $\mathfrak{F}$ [4, p. 517]. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. For any two classes $\mathfrak{F}$ and $\mathfrak{X}$ of groups, $\mathfrak{X}\mathfrak{F}$ is the class of groups G such that $G^{\mathfrak{F}} \in \mathfrak{X}$.

A *formation* is a class $\mathfrak{F}$ of groups with the following properties: (i) Every homomorphic image of any group in $\mathfrak{F}$ belongs to $\mathfrak{F}$; (ii) If $\mathfrak{F} \neq \emptyset$, then $G/G^{\mathfrak{F}} \in \mathfrak{F}$ for any group G. A formation $\mathfrak{F}$ is said to be: *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; *hereditary* if $H \in \mathfrak{F}$ whenever $H \leqslant G \in \mathfrak{F}$.

For any formation function $f : \mathbb{P} \to$ {formations of groups}, the symbol $LF(f)$ denotes the collection of all groups G such that either $G = 1$ or $G \neq 1$ and $G/C_G(H/K) \in f(p)$ for every chief factor H/K of G and every $p \in \pi(H/K)$. If $\mathfrak{F} = LF(f)$ for some formation function f, then f is said to be a local definition or a *local satellite* (Shemetkov) of $\mathfrak{F}$. Every non-empty saturated formation $\mathfrak{F}$ has a unique local satellite F with the following property: For any prime p, both $F(p) \subseteq \mathfrak{F}$ and $G \in F(p)$ whenever $G/O_p(G) \in F(p)$ (see [4, IV, Proposition 3.8]). Such a satellite is called the *canonical local satellite* of $\mathfrak{F}$.

A chief factor H/K of a group G is called $\mathfrak{F}$-*central in* G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. A normal subgroup N of G is said to be $\pi\mathfrak{F}$-*hypercentral in* G if either $N = 1$ or $N \neq 1$ and every chief factor of G below N of order divisible by at least one prime in π is $\mathfrak{F}$-central in G. The symbol $Z_{\pi\mathfrak{F}}(G)$ denotes the $\pi\mathfrak{F}$-*hypercentre* of G, that is, the product of all normal $\pi\mathfrak{F}$-hypercentral subgroups of G. It is clear that for the $\mathfrak{F}$-hypercentre $Z_{\mathfrak{F}}(G)$ of G (see [4, p. 389]) we have $Z_{\mathfrak{F}}(G) = Z_{\mathbb{P}\mathfrak{F}}(G)$. On the other limited case, when $\pi = \{p\}$, $Z_{p\mathfrak{F}}(G)$ is the product of all normal subgroups N of G such that every chief factor of G below N of order divisible by p is $\mathfrak{F}$-central.

A subgroup U of G is called $\mathfrak{F}$-*maximal* in G provided that (a) $U \in \mathfrak{F}$, and (b) if $U \leqslant V \leqslant G$ and $V \in \mathfrak{F}$, then $U = V$ [4, p. 288]. We denote the intersection of all $\mathfrak{F}$-maximal subgroups of G by $\mathrm{Int}_{\mathfrak{F}}(G)$. In the paper [3], Beidleman and Heineken characterized the subgroup $\mathrm{Int}_{\mathfrak{F}}(G)$ in the case when G is soluble and $\mathfrak{F}$ is a hereditary saturated formation. In this paper, as a development of some results in [12] and [13], we find some new properties and applications of the subgroup $\mathrm{Int}_{\mathfrak{F}}(G)$.

Baer [1] proved that $\mathrm{Int}_{\mathcal{N}}(G)$ coincides with the hypercentre $Z_\infty(G) = Z_{\mathcal{N}}(G)$ of G. Later, in [9], Sidorov proved that if $\mathfrak{F}$ is the class of all soluble groups G of nilpotent length $l(G) \leqslant r$ $(r \in \mathbb{N})$, then for each soluble group G, the equality $Z_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ holds. In the papers [12] and [13], the analogous results were obtained for the classes of all p-decomposable groups and of all groups G with $G' \leqslant F(G)$ in the universe of all groups. As one of our results in this paper, we shall also prove that the intersection of all maximal p-nilpotent subgroups of G coincides with the subgroup $Z_{p\mathcal{N}}(G)$. But in general, $Z_{\pi\mathfrak{F}}(G) \neq \mathrm{Int}_{\mathfrak{F}}(G)$, even when $\mathfrak{F}$ is the class of all supersoluble (all p-supersoluble, for any odd prime p) groups and G is soluble (see Theorem A and Remark 4.8 in Section 4).

Definition 1.1. Let $\mathfrak{X}$ be a non-empty class of groups and $\mathfrak{F} = LF(F)$ be a saturated formation, where F is the canonical local satellite of $\mathfrak{F}$. We say that $\mathfrak{F}$ satisfies the π-*boundary condition* (*the boundary condition* if $\pi = \mathbb{P}$) *in* $\mathfrak{X}$ if $G \in \mathfrak{F}$ whenever $G \in \mathfrak{X}$ and G is an $F(p)$-critical group for at least one $p \in \pi$.

We say that $\mathfrak{F}$ satisfies the π-*boundary condition* if $\mathfrak{F}$ satisfies the π-boundary condition in the class of all groups.

If $\mathfrak{F}$ is a non-empty formation with $\pi(\mathfrak{F}) = \emptyset$, then $\mathfrak{F} = (1)$, and therefore for any group G we have $Z_{\mathfrak{F}}(G) = 1 = \mathrm{Int}_{\mathfrak{F}}(G)$. In the other limited case, when $\mathfrak{F} = \mathfrak{G}$ is the class of all groups, we have $Z_{\mathfrak{F}}(G) = G = \mathrm{Int}_{\mathfrak{F}}(G)$. Similarly, if $\mathfrak{F} = \mathcal{S}$, then $Z_{\mathfrak{F}}(G) = G = \mathrm{Int}_{\mathfrak{F}}(G)$ for every soluble group G.

For the general case, we shall prove the following.

Theorem A. *Let $\mathfrak{F}$ be a hereditary saturated formation with $(1) \neq \mathfrak{F} \neq \mathfrak{G}$. Let $\pi \subseteq \pi(\mathfrak{F})$. Then the equality*

$$Z_{\pi\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$$

holds for each group G if and only if $\mathcal{N} \subseteq \mathfrak{F} = \mathfrak{G}_{\pi'}\mathfrak{F}$ and $\mathfrak{F}$ satisfies the π-boundary condition.

Note that $N(p) = \mathfrak{G}_p$, where N is the canonical local satellite of $\mathfrak{N}$. Hence every $N(p)$-critical group has prime order. This shows that $\mathfrak{N}$ satisfies the boundary condition and so the above-mentioned Baer's result is a corollary of Theorem A.

Theorem B. *Let $\mathfrak{F}$ be a hereditary saturated formation of soluble groups with $(1) \neq \mathfrak{F} \neq \mathfrak{S}$. Let $\pi \subseteq \pi(\mathfrak{F})$. Then the equality*

$$Z_{\pi\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$$

holds for each soluble group G if and only if $\mathfrak{N} \subseteq \mathfrak{F} = \mathfrak{S}_{\pi'}\mathfrak{F}$ and $\mathfrak{F}$ satisfies the π-boundary condition in the class of all soluble groups.

If for some classes $\mathfrak{F}$ and $\mathfrak{M}$ of groups we have $\mathfrak{F} \subseteq \mathfrak{M}$, then every $\mathfrak{F}$-maximal subgroup of a group is contained in some its $\mathfrak{M}$-maximal subgroup. Nevertheless, the following example shows that in general, $\mathrm{Int}_{\mathfrak{F}}(G) \nleqslant \mathrm{Int}_{\mathfrak{M}}(G)$.

Example 1.2. Let $\mathfrak{F} = \mathfrak{U}$ and $\mathfrak{M}$ be the class of all p-supersoluble groups, where $p > 2$. Let q be a prime dividing $p - 1$ and $G = P \rtimes (Q \rtimes C)$, where C is a group of order p, Q is a simple $\mathbb{F}_q G$-module which is faithful for C and P is a simple $\mathbb{F}_p G$-module which is faithful for $Q \rtimes C$. Then, clearly, $P = \mathrm{Int}_{\mathfrak{F}}(G)$ and $\mathrm{Int}_{\mathfrak{M}}(G) = 1$.

This example is a motivation for the following our result.

Theorem C. *Let $\mathfrak{F} \subseteq \mathfrak{M} = LF(M)$ be hereditary saturated formations with $\pi \subseteq \pi(\mathfrak{F})$, where M is the canonical local satellites of $\mathfrak{M}$.*

(a) *Suppose that $\mathfrak{N} \subseteq \mathfrak{M} = \mathfrak{G}_{\pi'}\mathfrak{M}$ and $\mathfrak{F}$ satisfies the π-boundary condition in $\mathfrak{M}$. Then the inclusion*

$$\mathrm{Int}_{\mathfrak{F}}(G) \leqslant \mathrm{Int}_{\mathfrak{M}}(G)$$

holds for each group G.

(b) *If every (soluble) $M(p)$-critical group belongs to $\mathfrak{F}$ for every $p \in \pi$, then $\mathfrak{N} \subseteq \mathfrak{M}$ and*

$$\mathrm{Int}_{\mathfrak{F}}(G) \leqslant Z_{\pi\mathfrak{M}}(G)$$

for every (soluble) group G.

Recall that a subgroup H of a group G is said to be $\mathfrak{F}$-*subnormal* (in the sense of Kegel [8]) or K-$\mathfrak{F}$-*subnormal* in G (see p. 236 in [2]) if either $H = G$ or there exists a chain of subgroups

$$H = H_0 < H_1 < \cdots < H_t = G$$

such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$ for all $i = 1, \ldots, t$.

For any group G, we write $\mathrm{Int}^*_{\mathfrak{F}}(G)$ to denote the intersection of all non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroups of G. The following theorem shows that for any hereditary saturated formation $\mathfrak{F}$ with $\mathfrak{N} \subseteq \mathfrak{F}$, the intersection of all non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroups of a group G coincides with $\mathrm{Int}_{\mathfrak{F}}(G)$.

Theorem D. *Let $\mathfrak{F}$ be a hereditary saturated formation containing all nilpotent groups. Then the equality*

$$\mathrm{Int}^*_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$$

holds for each group G.

We prove Theorems A, B, C and D in Section 3. In Section 4 we give some examples and discuss applications of these theorems.

All unexplained notation and terminology are standard. The reader is referred to [4,2,6] if necessary.

2. Preliminaries

In view of Proposition 3.16 in [4, IV], we have

Lemma 2.1. *Let $\mathfrak{F} = LF(F)$ be a hereditary saturated formation, where F is the canonical local satellite of $\mathfrak{F}$. Then for any prime p, the formation $F(p)$ is hereditary.*

We shall need some facts about the $\pi\mathfrak{F}$-hypercentre in our proofs.

Lemma 2.2. *Let $\mathfrak{F} = LF(F)$ be a saturated formation, where F is the canonical local satellite of $\mathfrak{F}$. Let $\pi \subseteq \pi(\mathfrak{F})$ and $\sigma = \pi(\mathfrak{F}) \setminus \pi$. Let N be a normal subgroup of G, and $A \leqslant G$.*

(1) *A chief factor H/K of G is $\mathfrak{F}$-central if and only if $G/C_G(H/K) \in F(p)$ for all primes p dividing $|H/K|$.*

(2) *Every G-chief factor of $Z_{\pi\mathfrak{F}}(G)$ of order divisible by at least one prime in π is $\mathfrak{F}$-central.*

(3) *$Z_{\pi\mathfrak{F}}(G)N/N \leqslant Z_{\pi\mathfrak{F}}(G/N)$.*

(4) *$Z_{\pi\mathfrak{F}}(A)N/N \leqslant Z_{\pi\mathfrak{F}}(AN/N)$.*

(5) *If $\mathfrak{F}$ is (normally) hereditary and H is a (normal) subgroup of G, then $Z_{\pi\mathfrak{F}}(H) \cap A \leqslant Z_{\pi\mathfrak{F}}(H \cap A)$.*

(6) *If $\mathfrak{G}_\sigma\mathfrak{F} = \mathfrak{F}$ and $G/Z_{\pi\mathfrak{F}}(G) \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

(7) *Suppose that $\mathfrak{F}$ is (normally) hereditary and let H be a (normal) subgroup of G. If $\mathfrak{G}_\sigma\mathfrak{F} = \mathfrak{F}$ and $H \in \mathfrak{F}$, then $Z_{\pi\mathfrak{F}}(G)H \in \mathfrak{F}$.*

Proof. (1) This assertion is well known (see for example Theorem 17.14 in [11] or Theorem 3.1.6 in [6]). Assertions (2) and (6) are evident.

(3) Let H/K be a chief factor of G such that $N \leqslant K < H \leqslant NZ_{\pi\mathfrak{F}}(G)$ and $|H/K|$ is divisible by at least one prime in π. Then H/K is G-isomorphic to the chief factor $H \cap Z_{\pi\mathfrak{F}}(G)/K \cap Z_{\pi\mathfrak{F}}(G)$ of G. Therefore H/K is $\mathfrak{F}$-central in G by (1) and (2). Consequently, $Z_{\pi\mathfrak{F}}(G)N/N \leqslant Z_{\pi\mathfrak{F}}(G/N)$.

(4) Let $f : A/A \cap N \to AN/N$ be the canonical isomorphism from $A/A \cap N$ onto AN/N. Then $f(Z_{\pi\mathfrak{F}}(A/A \cap N)) = Z_{\pi\mathfrak{F}}(AN/N)$ and

$$f\big(Z_{\pi\mathfrak{F}}(A)(A \cap N)/(A \cap N)\big) = Z_{\pi\mathfrak{F}}(A)N/N.$$

By (3) we have

$$Z_{\pi\mathfrak{F}}(A)(A \cap N)/(A \cap N) \leqslant Z_{\pi\mathfrak{F}}(A/A \cap N).$$

Hence $Z_{\pi\mathfrak{F}}(A)N/N \leqslant Z_{\pi\mathfrak{F}}(AN/N)$.

(5) First suppose that $\mathfrak{F}$ is hereditary. Let

$$1 = Z_0 < Z_1 < \cdots < Z_t = Z_{\pi\mathfrak{F}}(G)$$

be a chief series of G below $Z_{\pi\mathfrak{F}}(G)$ and $C_i = C_G(Z_i/Z_{i-1})$. Let q be a prime divisor of

$$|Z_i \cap H/Z_{i-1} \cap H| = \big|Z_{i-1}(Z_i \cap H)/Z_{i-1}\big|.$$

Suppose that q divides $|Z_i \cap H/Z_{i-1} \cap H|$. Then q **divides** $|Z_i/Z_{i-1}|$, so $G/C_i \in F(q)$ by (1). Hence $H/H \cap C_i \simeq C_iH/C_i \in F(q)$. But $H \cap C_i \leqslant C_H(Z_i \cap H/Z_{i-1} \cap H)$. Hence $H/C_H(Z_i \cap H/Z_{i-1} \cap H) \in F(q)$ for all primes q dividing $|Z_i \cap H/Z_{i-1} \cap H|$. Thus $Z_{\pi\mathfrak{F}}(G) \cap H \leqslant Z_{\pi\mathfrak{F}}(H)$ by (1) and (2). But then

$$Z_{\pi\mathfrak{F}}(H) \cap E = Z_{\pi\mathfrak{F}}(H) \cap (H \cap E) \leqslant Z_{\pi\mathfrak{F}}(H \cap E).$$

Similarly, one may prove the second assertion of (5).

(7) Since $H \in \mathfrak{F}$ we have

$$HZ_{\pi\mathfrak{F}}(G)/Z_{\pi\mathfrak{F}}(G) \simeq H/H \cap Z_{\pi\mathfrak{F}}(G) \in \mathfrak{F}$$

and

$$Z_{\pi\mathfrak{F}}(G) \leqslant Z_{\pi\mathfrak{F}}\big(Z_{\pi\mathfrak{F}}(G)H\big)$$

by (5). Hence $HZ_{\pi\mathfrak{F}}(G) \in \mathfrak{F}$ by (6).

The lemma is proved. □

The following lemma is evident (note only that statement (i) directly follows from [4, Theorem A.9.2(c)]).

Lemma 2.3. *Let $\mathfrak{F}$ be a hereditary saturated formation. Let $N \leqslant U \leqslant G$, where N is a normal subgroup of G.*

(i) *If $G/N \in \mathfrak{F}$ and V is a minimal supplement of N in G, then $V \in \mathfrak{F}$.*

(ii) *If U/N is an $\mathfrak{F}$-maximal subgroup of G/N, then $U = U_0N$ for some $\mathfrak{F}$-maximal subgroup U_0 of G.*

(iii) *If V is an $\mathfrak{F}$-maximal subgroup of U, then $V = H \cap U$ for some $\mathfrak{F}$-maximal subgroup H of G.*

The proofs of our theorems are based on the following general facts on the subgroup $\mathrm{Int}_{\mathfrak{F}}(G)$.

Lemma 2.4. *Let $\mathfrak{F}$ be a hereditary saturated formation, $\pi \subseteq \pi(\mathfrak{F})$ and $\sigma = \pi(\mathfrak{F}) \setminus \pi$. Let H, E be subgroups of G, N a normal subgroup of G and $I = \mathrm{Int}_{\mathfrak{F}}(G)$. Then*:

(a) $\mathrm{Int}_{\mathfrak{F}}(H)N/N \leqslant \mathrm{Int}_{\mathfrak{F}}(HN/N)$.

(b) $\mathrm{Int}_{\mathfrak{F}}(H) \cap E \leqslant \mathrm{Int}_{\mathfrak{F}}(H \cap E)$.

(c) *If $H/H \cap I \in \mathfrak{F}$, then $H \in \mathfrak{F}$.*

(d) *If $H \in \mathfrak{F}$, then $IH \in \mathfrak{F}$.*

(e) *If $N \leqslant I$, then $I/N = \mathrm{Int}_{\mathfrak{F}}(G/N)$.*

(f) $\mathrm{Int}_{\mathfrak{F}}(G/I) = 1$.

(g) *If $\mathfrak{G}_\sigma \mathfrak{F} = \mathfrak{F}$, then $Z_{\pi\mathfrak{F}}(G) \leqslant I$.*

Proof. Assertions (a)–(f) are proved in [12]. Now we prove (g). Let H be a subgroup of G such that $H \in \mathfrak{F}$. Then $HZ_{\pi\mathfrak{F}}(G)/Z_{\pi\mathfrak{F}}(G) \simeq H/H \cap Z_{\pi\mathfrak{F}}(G) \in \mathfrak{F}$ and $Z_{\pi\mathfrak{F}}(G) \leqslant Z_{\pi\mathfrak{F}}(HZ_{\pi\mathfrak{F}}(G))$ by Lemma 2.2(5). It follows from Lemma 2.2(5) that $HZ_{\pi\mathfrak{F}}(G) \in \mathfrak{F}$. Thus $Z_{\pi\mathfrak{F}}(G) \leqslant I$. □

The following lemma is a corollary of general results on f-hypercentral action (see [4, Chapter IV, Section 6]). For the reader's convenience, we give a direct proof.

Lemma 2.5. *Let $\mathfrak{F} = LF(F)$ be a saturated formation, where F is the canonical local satellite of $\mathfrak{F}$. Let E be a normal p-subgroup of G. If $E \leqslant Z_{\mathfrak{F}}(G)$, then $G/C_G(E) \in F(p)$.*

Proof. Let $1 = E_0 < E_1 < \cdots < E_t = E$ be a chief series of G below E. Let $C_i = C_G(E_i/E_{i-1})$ and $C = C_1 \cap \cdots \cap C_t$. Then $C_G(E) \leqslant C$ and so $C/C_G(E)$ is a p-group by Corollary 3.3 in [5, Chapter 5]. On the other hand, by Lemma 2.2(1), $G/C_i \in F(p)$, so $G/C \in F(p)$. Hence $G/C_G(E) \in F(p) = \mathfrak{G}_pF(p)$. The lemma is proved. □

Lemma 2.6. *Let $\mathfrak{F} = LF(F)$ and $\mathfrak{M}$ be saturated formations with $p \in \pi(\mathfrak{F})$ and $\mathfrak{F} \subseteq \mathfrak{M}$, where F is the canonical local satellite of $\mathfrak{F}$. Suppose that G is a group of minimal order in the set of all $F(p)$-critical groups $G \in \mathfrak{M}$ with $G \notin \mathfrak{F}$. Then $O_p(G) = 1 = \Phi(G)$ and $G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G.*

Proof. Let N be a minimal normal subgroup of G. First we show that $G/N \in \mathfrak{F} \cap \mathfrak{M}$. Indeed, since $G \in \mathfrak{M}$ and $\mathfrak{M}$ is a formation, $G/N \in \mathfrak{M}$. Suppose that $G/N \notin \mathfrak{F}$. Then $G/N \notin F(p)$ since $F(p) \subseteq \mathfrak{F}$.

On the other hand, for any maximal subgroup M/N of G/N we have $M/N \in F(p)$ since $F(p)$ is a formation and G is an $F(p)$-critical group. Thus G/N is an $F(p)$-critical group with $G/N \notin \mathfrak{M} \setminus \mathfrak{F}$, which contradicts the minimality of G. Hence $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is a saturated formation, $N = G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G and $\Phi(G) = 1$. Suppose that $N \leqslant O_p(G)$ and let M be a maximal subgroup of G such that $G = NM$. Then $G/N \simeq M/N \cap M \in F(p) = \mathfrak{G}_p F(p)$, and so $G \leqslant F(p) \subseteq \mathfrak{F}$. This contradiction shows that $O_p(G) = 1$. The lemma is proved. $\square$

Lemma 2.7. *Let $\mathfrak{F}$ be a formation, H and E be subgroups of a group G, where H is K-$\mathfrak{F}$-subnormal in G. Then:*

(i) *$H \cap E$ is K-$\mathfrak{F}$-subnormal in E (see Theorem 6.1.7 in [2]).*

(ii) *If E is normal in G, then HE/E is K-$\mathfrak{F}$-subnormal in G/E (see Theorem 6.1.6 in [2]).*

Lemma 2.8. *Let $\mathfrak{F}$ be a hereditary saturated formation. Let $N \leqslant U \leqslant G$, where N is a normal subgroup of G.*

(i) *If U/N is a non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup of G/N, then $U = U_0N$ for some non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup U_0 of G.*

(ii) *If V is a non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup of U, then $V = H \cap U$ for some non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup H of G.*

Proof. (i) By Lemma 2.3(ii), there is an $\mathfrak{F}$-maximal subgroup U_0 of G such that $U = U_0N$. Since U/N is non-K-$\mathfrak{F}$-subnormal in G/N, U_0 is not K-$\mathfrak{F}$-subnormal in G by Lemma 2.7(ii).

(ii) By Lemma 2.3(iii), for some $\mathfrak{F}$-maximal subgroup H of G we have $V = H \cap U$. Since V is a non-K-$\mathfrak{F}$-subnormal in U, H is not K-$\mathfrak{F}$-subnormal in G by Lemma 2.7(i). $\square$

Lemma 2.9. *Let $\mathfrak{F} = LF(F)$ be a non-empty saturated formation, where F is the canonical local satellite of $\mathfrak{F}$.*

(1) *If $\mathfrak{F} = \mathfrak{G}_p\mathfrak{F}$ for some prime p, then $F(p) = \mathfrak{F}$.*

(2) *If $\mathfrak{F} = \mathfrak{N}\mathfrak{H}$ for some non-empty formation $\mathfrak{H}$, then $F(p) = \mathfrak{G}_p\mathfrak{H}$ for all primes p.*

Proof. (1) Since $F(p) \subseteq \mathfrak{F}$, we need only prove that $\mathfrak{F} \subseteq F(p)$. Suppose that this is false and let A be a group of minimal order in $\mathfrak{F} \diagdown F(p)$. Then $A^{F(p)}$ is a unique minimal normal subgroup of A since $F(p)$ is a formation. Moreover, $O_p(A) = 1$ since $F(p) = \mathfrak{G}_p F(p)$. Let $G = C_p \wr A = K \rtimes A$ where K is the base group of the regular wreath G. Then $K = O_{p',p}(G)$ and $G \in \mathfrak{F} = \mathfrak{G}_p\mathfrak{F}$. Hence $A \simeq G/K = G/O_{p',p}(G) \in F(p)$, a contradiction. Thus $F(p) = \mathfrak{F}$.

(2) The inclusion $F(p) \subseteq \mathfrak{G}_p\mathfrak{H}$ is evident. The inverse inclusion can be proved similarly as the inclusion $\mathfrak{F} \subseteq F(p)$ in the proof of (1). $\square$

We will also use in our proofs the following well-known elementary fact (see for example, [11, Lemma 18.8] or [6, Lemma 3.5.13]).

Lemma 2.10. *If $O_p(G) = 1$ and G has a unique minimal normal subgroup, then there exists a simple $\mathbb{F}_pG$-module which is faithful for G.*

3. Proofs of the theorems

Proof of Theorem C. (a) Suppose that this assertion is false and let G be a counterexample with minimal order. Let $I = \mathrm{Int}_{\mathfrak{F}}(G)$ and $I_1 = \mathrm{Int}_{\mathfrak{M}}(G)$. Then $1 < I < G$ and $I_1 \neq G$. Let L be a minimal normal subgroup of G contained in I and $C = C_G(L)$. Then $\pi(L) \subseteq \pi(\mathfrak{F})$.

(1) *$IN/N \leqslant \mathrm{Int}_{\mathfrak{F}}(G/N) \leqslant \mathrm{Int}_{\mathfrak{M}}(G/N)$ for any non-identity normal subgroup N of G.*

Indeed, by Lemma 2.4(a), we have $IN/N \leqslant \mathrm{Int}_{\mathfrak{F}}(G/N)$. On the other hand, by the choice of G, $\mathrm{Int}_{\mathfrak{F}}(G/N) \leqslant \mathrm{Int}_{\mathfrak{M}}(G/N)$.

(2) *$L \nleqslant I_1$; in particular, the order of L is divisible by some prime $p \in \pi$.*

Suppose that $L \leqslant I_1$. Then $I_1/L = \mathrm{Int}_{\mathfrak{M}}(G/L)$ by Lemma 2.4(e). But by (1), $IL/L = I/L \leqslant \mathrm{Int}_{\mathfrak{F}}(G/L) \leqslant \mathrm{Int}_{\mathfrak{M}}(G/L)$. Hence $I/L \leqslant I_1/L$ and so $I \leqslant I_1$, a contradiction. Thus $L \nleqslant I_1$. This means that there exists an $\mathfrak{M}$-maximal subgroup M of G such that $L \nleqslant M$. Suppose that L is a π'-group.

Then $LM \in \mathfrak{G}_{\pi'}\mathfrak{M} = \mathfrak{M}$, which contradicts the maximality of M. Hence the order of L is divisible by some prime $p \in \pi$.

(3) *If $L \leqslant M < G$, then $L \leqslant \mathrm{Int}_{\mathfrak{M}}(M)$.*

By Lemma 2.4(b), $L \leqslant I \cap M \leqslant \mathrm{Int}_{\mathfrak{F}}(M)$. But since $|M| < |G|$, $\mathrm{Int}_{\mathfrak{F}}(M) \leqslant \mathrm{Int}_{\mathfrak{M}}(M)$ by the choice of G. Hence $L \leqslant \mathrm{Int}_{\mathfrak{M}}(M)$.

(4) *$G = LU$ for any $\mathfrak{M}$-maximal subgroup U of G not containing L. In particular, $G/L \in \mathfrak{M}$.*

Indeed, suppose that $LU \neq G$. Then by (3), $L \leqslant \mathrm{Int}_{\mathfrak{M}}(LU)$, which implies that $LU \in \mathfrak{M}$ by Lemma 2.4(c). This contradicts the $\mathfrak{M}$-maximality of U. Hence we have (4).

(5) *$C_G(L) \cap U = U_G = 1$ for any $\mathfrak{M}$-maximal subgroup U of G not containing L.*

Since $C_G(L)$ is normal in G and $G = LU$ by (4), $U_G = C_G(L) \cap U$. Assume that $U_G \neq 1$. Let $U/U_G \leqslant W/U_G$, where W/U_G is an $\mathfrak{M}$-maximal subgroup of G/U_G. Then by (1), $LU_G/U_G \leqslant W/U_G$. Hence $G = LU \leqslant W$ by (4), which means that $G/U_G = W/U_G \in \mathfrak{M}$. But by (4), $G/L \in \mathfrak{M}$. Therefore $G \simeq G/L \cap U_G \in \mathfrak{M}$, and consequently $I = G$, a contradiction. Hence (5) holds.

The final contradiction for (a). Since $L \nleqslant I_1$ by (2), there is an $\mathfrak{M}$-maximal subgroup M of G such that $L \nleqslant M$. But then $G = LM$ by (4). Since $L \leqslant I$ and $G \notin \mathfrak{F}$, $M \notin \mathfrak{F}$ by Lemma 2.4(d). Let H be an $\mathfrak{F}$-critical subgroup of M, V a maximal subgroup of H. We show that $V \in F(p)$. By Lemma 2.4(d), $D = LV \in \mathfrak{F}$. Hence $D/O_{p',p}(D) \in F(p)$. First assume that L is a non-abelian group. Then, since p divides $|L|$, $O_{p',p}(D) \cap L = 1$. Hence $O_{p',p}(D) \leqslant C_G(L)$ and $O_{p',p}(D) \cap V = 1$ by (5). Since $\mathfrak{F}$ is hereditary, $F(p)$ is hereditary by Lemma 2.1. Therefore $O_{p',p}(D)V/O_{p',p}(D) \simeq V \in F(p)$. Now assume that L is an abelian p-group. Then $L \leqslant O_{p',p}(D)$ and $O_{p',p}(D) = L(O_{p',p}(D) \cap V)$. Hence $O_{p'}(D) \leqslant M \cap C_G(L) = 1$. It follows that $O_{p',p}(D) = O_p(D)$. Therefore $D/O_p(D) \in F(p) = \mathfrak{G}_p F(p)$, which implies that $D \in F(p)$ and so $V \in F(p)$. Therefore H is an $F(p)$-critical group. Since $\mathfrak{M}$ is hereditary and $M \in \mathfrak{M}$, $H \in \mathfrak{M}$. But then $H \in \mathfrak{F}$ since $\mathfrak{F}$ satisfies the π-boundary condition in $\mathfrak{M}$ by hypothesis. This contradiction completes the proof of (a).

(b) Suppose that every $M(p)$-critical group G belongs to $\mathfrak{F}$ for every $p \in \pi$. First we show that $\mathfrak{N} \subseteq \mathfrak{M}$. Assume that this is false and let C_q be a group of prime order q with $C_q \notin \mathfrak{M}$. Let $p \in \pi$. Then C_q is $M(p)$-critical and so $C_q \in \mathfrak{F} \subseteq \mathfrak{M}$ by the hypothesis. This contradiction shows that $\mathfrak{N} \subseteq \mathfrak{M}$.

Now we show that $\mathrm{Int}_{\mathfrak{F}}(G) \leqslant Z_{\pi\mathfrak{M}}(G)$ for every group G. Suppose that this assertion is false and let G be a counterexample with minimal order. Let $I = \mathrm{Int}_{\mathfrak{F}}(G)$ and $Z = Z_{\pi\mathfrak{M}}(G)$. Then $1 < I < G$ and $Z \neq G$. Let N be a minimal normal subgroup of G and L a minimal normal subgroup of G contained in I. Then $\pi(L) \leqslant \pi(\mathfrak{F})$. We proceed via the following steps.

(1) $IN/N \leqslant \mathrm{Int}_{\mathfrak{F}}(G/N) \leqslant Z_{\pi\mathfrak{M}}(G/N)$.

Indeed, by Lemma 2.4(a), we have $IN/N \leqslant \mathrm{Int}_{\mathfrak{F}}(G/N)$. On the other hand, by the choice of G, $\mathrm{Int}_{\mathfrak{F}}(G/N) \leqslant Z_{\pi\mathfrak{M}}(G/N)$.

(2) *$L \nleqslant Z$; in particular, the order of L is divisible by some prime $p \in \pi$.*

Suppose that $L \leqslant Z$. Then, clearly, $Z/L = Z_{\pi\mathfrak{M}}(G/L)$, and $I/L = \mathrm{Int}_{\mathfrak{F}}(G/L)$ by Lemma 2.4(e). But by (1), $\mathrm{Int}_{\mathfrak{F}}(G/L) \leqslant Z_{\pi\mathfrak{M}}(G/L)$. Hence $I/L \leqslant Z/L$. Consequently, $I \leqslant Z$, a contradiction.

(3) *If $L \leqslant M < G$, then $L \leqslant Z_{\mathfrak{M}}(M)$.*

By Lemma 2.4(b), $L \leqslant I \cap M \leqslant \mathrm{Int}_{\mathfrak{F}}(M)$. But since $|M| < |G|$, we have that $\mathrm{Int}_{\mathfrak{F}}(M) \leqslant Z_{\pi\mathfrak{M}}(M)$ by the choice of G. Hence $L \leqslant Z_{\pi\mathfrak{M}}(M)$ and so $L \leqslant Z_{\mathfrak{M}}(M)$ since the order of L is divisible by some prime $p \in \pi$ by (2).

(4) *$L = N$ is the unique minimal normal subgroup of G.*

Suppose that $L \neq N$. Then by (1), $NL/N \leqslant Z_{\pi\mathfrak{M}}(G/N)$. Hence from the G-isomorphism $NL/N \simeq L$ we obtain $L \leqslant Z$, which contradicts (2).

(5) $L \nleqslant \Phi(G)$.

Suppose that $L \leqslant \Phi(G)$. Then L is a p-group by (2). Let $C = C_G(L)$ and M be any maximal subgroup of G. Then $L \leqslant M$. Hence $L \leqslant Z_{\mathfrak{M}}(M)$ by (3), so $M/M \cap C \in M(p)$ by Lemmas 2.1(1) and 2.5. If $C \nleqslant M$, then $G/C = CM/C \simeq M/M \cap C \in M(p)$. This implies that $L \leqslant Z$, which contradicts (2). Hence $C \leqslant M$ for all maximal subgroups M of G. It follows that C is nilpotent. Then in view of (4), C is a p-group since C is normal in G. Hence for every maximal subgroup M of G we have $M \in \mathfrak{G}_p M(p) = M(p)$. But since $M(p) \subseteq \mathfrak{M}$, $G \notin M(p)$ (otherwise $G \in \mathfrak{M}$ and so $G = Z$). This shows that G is an $M(p)$-critical group. Therefore $G \in \mathfrak{F}$ by the hypothesis. But since $\mathfrak{F} \subseteq \mathfrak{M}$, we have $G \in \mathfrak{M}$ and so $G = Z$, a contradiction. Thus (5) holds.

(6) $C = C_G(L) \leqslant L$ (this follows from (4), (5) and Theorem 15.6 in [4, Chapter A]).

(7) *If $L \leqslant M < G$, then $M \in M(p)$.*

First by (3), $L \leqslant Z_{\mathcal{M}}(M)$. If $L = C$, then $M/L = M/M \cap C \in M(p)$ by Lemma 2.5, which implies that $M \in \mathcal{G}_p M(p) = M(p)$ since L is a p-group by (2). Now suppose that L is a non-abelian group. Let $1 = L_0 < L_1 < \cdots < L_n = L$ be a chief series of M below L. Let $C_i = C_M(L_i/L_{i-1})$ and $C_0 = C_1 \cap \cdots \cap C_n$. Since $L \leqslant Z_{\mathcal{M}}(M)$, $M/C_i \in M(p)$ for all $i = 1, \ldots, n$. It follows that $M/C_0 \in M(p)$. Since $C = 1$ by (4) and (6), for any minimal normal subgroup R of M we have $R \leqslant L$. Suppose that $C_0 \neq 1$ and let R be a minimal normal subgroup of M contained in C_0. Then $R \leqslant L$ and $R \leqslant C_M(H/K)$ for each chief factor H/K of M. Thus $R \leqslant F(M)$ is abelian and hence L is abelian. This contradiction shows that $C_0 = 1$. Consequently, $M \in M(p)$.

(8) *There exists a subgroup U of G such that $U \in \mathcal{F}$ and $LU = G$.*

Indeed, suppose that every maximal subgroup of G not containing L belongs to $M(p)$. Then by (7), G is an $M(p)$-critical group. Hence $G \in \mathcal{F}$ by the hypothesis. But then $I = G$, a contradiction. Hence there exists a maximal subgroup M of G such that $G = LM$ and $M \notin M(p)$. Take an $M(p)$-critical subgroup U of M. Then in view of (7), $LU = G$ and $U \in \mathcal{F}$ by the hypothesis.

(9) *The final contradiction for* (b).

Since $L \leqslant I$ and $G/L = UL/L \simeq U/U \cap L \in \mathcal{F}$ by (8), it follows from Lemma 2.4(c) that $G \in \mathcal{F}$ and so $G = I$. The final contradiction shows that $\mathrm{Int}_{\mathcal{F}}(G) \leqslant Z_{\pi\mathcal{M}}(G)$ for every group G. The second assertion of (b) can be proved similarly. The theorem is proved. □

Proofs of Theorems A and B. Since $Z_{\mathcal{F}}(G) \leqslant \mathrm{Int}_{\mathcal{F}}(G)$ by Lemma 2.4(g), the sufficiency is a special case, when $\mathcal{F} = \mathcal{M}$, of Theorem C(b). Now suppose that the equality $Z_{\pi\mathcal{F}}(G) = \mathrm{Int}_{\mathcal{F}}(G)$ holds for each (soluble) group G.

First we show that $\mathcal{N} \subseteq \mathcal{F}$. Let F be the canonical local satellite of $\mathcal{F}$. Suppose that for some group C_q of prime order q we have $C_q \notin \mathcal{F}$. Let $p \in \pi$ and $G = PC_q$, where P is a simple $\mathbb{F}_p C_q$-module P which is faithful for C_q. Then $P = \mathrm{Int}_{\mathcal{F}}(G)$ and $Z_{\mathcal{F}}(G) = 1$ since $F(p) \subseteq \mathcal{F}$. This contradiction shows that $\mathcal{N} \subseteq \mathcal{F}$.

Now we show that $\mathcal{G}_{\pi'}\mathcal{F} = \mathcal{F}$ ($\mathcal{S}_{\pi'}\mathcal{F} = \mathcal{F}$, respectively). The inclusion $\mathcal{F} \subseteq \mathcal{S}_{\pi'}\mathcal{F}$ is evident. Suppose that $\mathcal{G}_{\pi'}\mathcal{F} \nsubseteq \mathcal{F}$ ($\mathcal{S}_{\pi'}\mathcal{F} \nsubseteq \mathcal{F}$) and let G be a group of minimal order in $\mathcal{G}_{\pi'}\mathcal{F} \setminus \mathcal{F}$ (in $\mathcal{S}_{\pi'}\mathcal{F} \setminus \mathcal{F}$, respectively). Then $G^{\mathcal{F}}$ is the unique minimal normal subgroup of G and $G^{\mathcal{F}}$ is a π'-group. Hence

$$G^{\mathcal{F}} \leqslant Z_{\pi\mathcal{F}}(G) = \mathrm{Int}_{\mathcal{F}}(G).$$

It follows from Lemma 2.4(c) that $G \in \mathcal{F}$. This contradiction shows that $\mathcal{G}_{\pi'}\mathcal{F} = \mathcal{F}$ ($\mathcal{S}_{\pi'}\mathcal{F} = \mathcal{F}$, respectively).

Finally, we show that $\mathcal{F}$ satisfies the π-boundary condition (the π-boundary condition in the class $\mathcal{S}$, respectively). Suppose that this is false. Then for some $p \in \pi$, the set of all (soluble) $F(p)$-critical groups A with $A \notin \mathcal{F}$ is non-empty. Let us choose in this set a group A with minimal $|A|$. Then by Lemma 2.6, $A^{\mathcal{F}}$ is the unique minimal normal subgroup of G and $O_p(A) = 1 = \Phi(A)$. Hence by Lemma 2.10, there exists a simple $\mathbb{F}_p A$-module P which is faithful for A. Let $G = P \rtimes A$ and M be any maximal subgroup of G. If $P \nleqslant M$, then $M \simeq G/P \simeq A \notin \mathcal{F}$. On the other hand, if $P \leqslant M$, then $M = M \cap PA = P(M \cap A)$, where $M \cap A$ is a maximal subgroup of A. Hence $M \cap A \in F(p)$ and so $M \in \mathcal{G}_p F(p) = F(p) \subseteq \mathcal{F}$. Therefore P is contained in the intersection of all $\mathcal{F}$-maximal subgroups of G. Then $P \leqslant Z_{\pi\mathcal{F}}(G)$ by our assumption on $\mathcal{F}$. It follows that $A \simeq G/P = G/C_G(P) \in F(p) \subseteq \mathcal{F}$ by Lemma 2.2(1) and Lemma 2.5. But this contradicts the choice of A. Therefore $\mathcal{F}$ satisfies the π-boundary condition ($\mathcal{F}$ satisfies the π-boundary condition in the class $\mathcal{S}$). The theorems are proved. □

In view of Theorems A, B and C we have

Corollary 3.1. *Let $\mathcal{F}$ be a hereditary saturated formation with $(1) \neq \mathcal{F} \neq \mathcal{G}$. Then the equality $\mathrm{Int}_{\mathcal{F}}(G) = Z_{\mathcal{F}}(G)$ holds for each group G if and only if $\mathcal{N} \subseteq \mathcal{F}$ and $\mathcal{F}$ satisfies the boundary condition.*

Corollary 3.2. *Let $\mathcal{F}$ be a hereditary saturated formation of soluble groups with $(1) \neq \mathcal{F} \neq \mathcal{S}$. Then the equality $\mathrm{Int}_{\mathcal{F}}(G) = Z_{\mathcal{F}}(G)$ holds for each soluble group G if and only if $\mathcal{N} \subseteq \mathcal{F}$ and $\mathcal{F}$ satisfies the boundary condition in the class $\mathcal{S}$.*

Note that Corollary 3.2 also follows directly from [3, Main Theorem].

Proof of Theorem D. We will prove the theorem by induction on $|G|$. If $G \in \mathfrak{F}$, then

$$\mathrm{Int}^*_{\mathfrak{F}}(G) = G = \mathrm{Int}_{\mathfrak{F}}(G).$$

We may, therefore, assume that $G \notin \mathfrak{F}$. Let $I = \mathrm{Int}_{\mathfrak{F}}(G)$, $I^* = \mathrm{Int}^*_{\mathfrak{F}}(G)$ and N be a minimal normal subgroup of G. Then $I \leqslant I^*$. Hence we may assume that $I^* \neq 1$.

(1) $I^*N/N \leqslant \mathrm{Int}^*_{\mathfrak{F}}(G/N)$.

If U/N is a non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup of G/N, then for some non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup U_0 of G we have $U = U_0N$ by Lemma 2.8(i). Let

$$\mathrm{Int}^*_{\mathfrak{F}}(G/N) = U_1/N \cap \cdots \cap U_t/N,$$

where U_i/N is a non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup of G/N for all $i = 1, \ldots, t$. Let V_i be a non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup of G such that $U_i = V_iN$. Then $I^* \leqslant V_1 \cap \cdots \cap V_t$. Hence $I^*N/N \leqslant \mathrm{Int}^*_{\mathfrak{F}}(G/N)$.

(2) *If $N \leqslant I^*$, then $\mathrm{Int}^*_{\mathfrak{F}}(G/N) = I^*/N$.*

By Lemma 2.8(i), it is enough to prove that if U is a non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup of G, then U/N is a non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup of G/N. Let $U/N \leqslant X/N$, where X/N is a non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup of G/N. By Lemma 2.8(i), $X = U_0N$ for some non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup U_0 of G. But since $N \leqslant U_0$, $U/N \leqslant U_0/N$ and so $U = U_0$. Thus $U/N = X/N$.

(3) *$I^* \cap H \leqslant \mathrm{Int}^*_{\mathfrak{F}}(H)$ for any subgroup H of G.*

Let V be an arbitrary non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup of H. Then $V = H \cap U$ for some non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroup U of G by Lemma 2.8(ii). Thus there are non-K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroups $U_1, \ldots, U_t$ of G such that

$$\mathrm{Int}^*_{\mathfrak{F}}(H) = U_1 \cap \cdots \cap U_t \cap H.$$

This induces that $I^* \cap H \leqslant \mathrm{Int}^*_{\mathfrak{F}}(H)$.

(4) *If $E = DV$ for some normal subgroup D of G contained in I^* and some K-$\mathfrak{F}$-subnormal subgroup $V \in \mathfrak{F}$ of G, then $E \in \mathfrak{F}$.*

First note that $R \leqslant \mathrm{Int}^*_{\mathfrak{F}}(E)$ by (3). On the other hand, by Lemma 2.7(i), V is a K-$\mathfrak{F}$-subnormal subgroup of E. Hence we need only consider the case when $G = E$. Assume that $G \notin \mathfrak{F}$. Let R be any minimal normal subgroup of G. Then $(DR/R)(VR/R) = G/R$, where $DR/R \leqslant \mathrm{Int}^*_{\mathfrak{F}}(G/R)$ by (1), and $VR/R \simeq V/V \cap R \in \mathfrak{F}$ is a K-$\mathfrak{F}$-subnormal subgroup of G/R. Hence by induction we have $G/R \in \mathfrak{F}$. This implies that R is the only minimal normal subgroup of G and so $R = G^{\mathfrak{F}} \leqslant I^*$. Let W be a minimal supplement of R in G. Then $W \in \mathfrak{F}$ by Lemma 2.3(i). Let $W \leqslant M$, where M is an $\mathfrak{F}$-maximal subgroup of G. If M is not K-$\mathfrak{F}$-subnormal in G, then $R \leqslant M$. Thus $G = RW = RM = M \in \mathfrak{F}$, a contradiction. This shows that M is K-$\mathfrak{F}$-subnormal in G. But then there is a proper subgroup X of G such that $M \leqslant X$ and either X is normal in G or $R = G^{\mathfrak{F}} \leqslant X$. In both of these cases, we have that $G = RM = RX = X < G$, a contradiction. Hence we have (4).

Conclusion. Let R be a minimal normal subgroup of G contained in I^*. If $R \leqslant I$, then $I/R = \mathrm{Int}_{\mathfrak{F}}(G/R)$ by Lemma 2.4(e), and $I^*/R = \mathrm{Int}^*_{\mathfrak{F}}(G/R)$ by (2). Therefore by induction, $\mathrm{Int}^*_{\mathfrak{F}}(G/R) = \mathrm{Int}_{\mathfrak{F}}(G/R)$. It follows that $I = I^*$.

Finally, suppose that $R \nleqslant I$. Then $R \nleqslant U$ for some $\mathfrak{F}$-maximal subgroup U of G. Let $E = RU$. Then $R \leqslant \mathrm{Int}^*_{\mathfrak{F}}(E)$ by (3). On the other hand, it is clear that U is a K-$\mathfrak{F}$-subnormal subgroup of G. Hence by (4), $E \in \mathfrak{F}$. But then $E = U$, which implies $R \leqslant U$, a contradiction. The theorem is proved. $\square$

4. Applications and remarks

Applications of Theorems A, B *and* D. We say that $\mathfrak{F}$ *satisfies the p-boundary condition* if $\mathfrak{F}$ satisfies the $\{p\}$-boundary condition in the class of all groups.

Lemma 4.1. *Let $\mathfrak{F} = LF(F)$, where F is the canonical local satellite of $\mathfrak{F}$. Suppose that for some prime p we have $F(p) = \mathfrak{F}$. Then $\mathfrak{F}$ does not satisfy the p-boundary condition.*

Proof. Indeed, in this case every $\mathfrak{F}$-critical group is also $F(p)$-critical. □

A group G is called π-closed if G has a normal Hall π-subgroup.

Proposition 4.2. *The formation $\mathfrak{F}$ of all π-closed groups satisfies the π'-boundary condition, but $\mathfrak{F}$ does not satisfy the p-boundary condition for any $p \in \pi$.*

Proof. Let $\mathfrak{F} = \mathfrak{G}_\pi \mathfrak{G}_{\pi'}$ be the formation of all π-closed groups, F the canonical local satellite of $\mathfrak{F}$. Then $F(p) = \mathfrak{F}$ for all $p \in \pi$, and $F(p) = \mathfrak{G}_{\pi'}$ for all primes $p \in \pi'$ by Theorem 3.1.20 in [6]. Hence $\mathfrak{F}$ satisfies the π'-boundary condition and does not satisfy the p-boundary condition for any $p \in \pi$ by Lemma 4.1. □

Lemma 4.3. *Let $\{\mathfrak{F}_i \mid i \in I\}$ be any set of non-empty saturated formations and $\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}_i$.*

(1) *If for each $i \in I$, $\mathfrak{F}_i$ satisfies the p-boundary condition, then $\mathfrak{F}$ satisfies the p-boundary condition.*

(2) *Suppose that $I = \{1, 2\}$, F_i is the canonical local satellite of $\mathfrak{F}_i$ and that there is a set π of primes satisfying the following conditions*:

(a) *$\mathfrak{F}_1$ satisfies the π-boundary condition, and for any $p \in \pi$, we have $F_1(p) \subseteq \mathfrak{F}_2 = F_2(p)$ and every $F_1(p)$-critical group belongs to $\mathfrak{F}_2$.*

(b) *$\mathfrak{F}_2$ satisfies the π'-boundary condition, and for any $p \in \pi'$, we have $F_2(p) \subseteq \mathfrak{F}_1 = F_1(p)$ and every $F_2(p)$-critical group belongs to $\mathfrak{F}_1$.*

Then $\mathfrak{F}$ satisfies the boundary condition.

Proof. (1) Let F_i be the canonical local satellite of $\mathfrak{F}_i$ and F the canonical local satellite of $\mathfrak{F}$. If $f(p) = \bigcap_{i \in I} F_i(p)$, then $F(p) = \mathfrak{G}_p f(p)$ by Theorem 3.3 in [10, Chapter 1]. Now let G be any $F(p)$-critical group, $i \in I$. Since $F(p) \subseteq F_i(p)$, all maximal subgroup of G belongs to $F_i(p)$. Hence $G \in \mathfrak{F}_i$ since $F_i(p) \subseteq \mathfrak{F}_i$ and $\mathfrak{F}_i$ satisfies the p-boundary condition. This implies that $G \in \mathfrak{F}$ and therefore $\mathfrak{F}$ satisfies the p-boundary condition.

(2) In this case, $F(p) = F_1(p)$ for all $p \in \pi$ and $F(p) = F_2(p)$ for all $p \in \pi'$, where F is the canonical local satellite of $\mathfrak{F}$. Hence if $p \in \pi$ and G is an $F(p)$-critical group, then $G \in \mathfrak{F}$ by hypothesis (a). This shows that $\mathfrak{F}$ satisfies the π-boundary condition. Similarly we see that $\mathfrak{F}$ satisfies the π'-boundary condition. □

A group G is called p-decomposable if there exists a subgroup H of G such that $G = P \times H$ for some (and hence the unique) Sylow p-subgroup P of G.

Corollary 4.4. *The formation of all p-decomposable groups satisfies the boundary condition.*

Proof. Let $\mathfrak{F}$ be the formation of all p-decomposable groups. Then $\mathfrak{F} = \mathfrak{G}_{p'} \mathfrak{G}_p \cap \mathfrak{G}_p \mathfrak{G}_{p'}$. Hence the assertion follows from Proposition 4.2 and Lemma 4.3(2). □

From Corollary 4.4 and Theorem A we get

Corollary 4.5. *Let D be the intersection of all maximal p-decomposable subgroups of G. Then D is the largest normal subgroup of G satisfying $D = O_{p'}(D) \times O_p(D)$, and G induces the trivial automorphisms group on every chief factor of G below $O_p(D)$ and a π'-group of automorphisms on every chief factor of G below $O_{p'}(D)$.*

Since a p-nilpotent group is p'-closed, the following result directly follows from Proposition 4.2.

Corollary 4.6. *The formation of all p-nilpotent groups satisfies the p-boundary condition.*

From Corollary 4.6 and Theorem A, we have

Corollary 4.7. *Let D be the intersection of all maximal p-nilpotent subgroups of a group G. Then D is the largest normal subgroup of G satisfying $O_{p'}(D) = O_{p'}(G)$, and $D/O_{p'}(G) \leqslant Z_\infty(G/O_{p'}(G))$.*

Note that another proof of Corollary 4.7 was obtained in the paper [3].

Remark 4.8. If $\pi \neq \{2\}$, then the formation $\mathfrak{F}$ of all π-supersoluble groups does not satisfy the π-boundary condition in the class of all soluble groups. Indeed, let F be the canonical local satellite of $\mathfrak{F}$. Then $F(p) = \mathfrak{N}_p\mathcal{A}(p-1)$, where $\mathcal{A}(p-1)$ is the formation of all abelian groups of exponent dividing $p-1$ [4, p. 358], for all $p \in \pi$ and $F(p) = \mathfrak{F}$ for all primes $q \notin \pi$ (see Example 3.4(e) and Theorem 4.8 in [4, Chapter IV]). Let $2 \neq p \in \pi$ and q be any prime with q divides $p-1$. Let $G = Q \rtimes C$, where C is a group of order p and Q is an $\mathbb{F}_qC$-module which is faithful for C. Then G is soluble, but G is a non-supersoluble $F(p)$-critical groups.

Proposition 4.9. *Let $\mathfrak{F} = \mathfrak{N}\mathfrak{L}$.*

(i) *If $\mathfrak{L}$ is a hereditary saturated formation satisfying the boundary condition in the class of all soluble groups, then $\mathfrak{F}$ satisfies the boundary condition in the class of all soluble groups.*

(ii) *If $\mathfrak{L}$ is a formation of nilpotent groups with $\pi(\mathfrak{L}) = \mathbb{P}$, then $\mathfrak{F}$ satisfies the boundary condition.*

Proof. Let F be the canonical local satellite of $\mathfrak{F}$. Then by Lemma 2.9(2), $F(p) = \mathfrak{G}_p\mathfrak{L}$ for all primes p. Assume that $\mathfrak{F}$ does not satisfy the boundary condition ($\mathfrak{F}$ does not satisfy the boundary condition in the class of all soluble groups). Then for some prime p, the set of all $F(p)$-critical (soluble) groups A with $A \notin \mathfrak{F}$ is non-empty. Let G be a group of minimal order in this set. Then $L = G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G and $O_p(G) = 1 = \Phi(G)$ by Lemma 2.6.

First suppose that G is soluble. Then $L = C_G(L)$ is a q-group for some prime $q \neq p$ and $G = L \rtimes M$ for some maximal subgroup M of G with $O_q(M) = 1$ by [4, Chapter A, Theorem 15.6]. Let M_1 be any maximal subgroup of M. Then $LM_1 \in F(p)$, and so $LM_1 \in \mathfrak{L}$ since $L = C_G(L)$ is a q-group.

Suppose that $\mathfrak{L}$ satisfies the condition of (i) and let L be the canonical local satellite of $\mathfrak{L}$. Since $LM_1 \in \mathfrak{L}$,

$$LM_1/O_{q',q}(LM_1) = LM_1/O_q(LM_1) = LM_1/LO_q(M_1) \simeq M_1/M_1 \cap LO_q(M_1) \in L(q) = \mathfrak{G}_pL(q).$$

Hence every maximal subgroup of M belongs to $L(q)$. Since $\mathfrak{L}$ satisfies the boundary condition in the class of all soluble groups, we obtain that $M \in \mathfrak{L}$ and consequently $G \in \mathfrak{F}$. This contradiction completes the proof of (i).

Suppose that $\mathfrak{L}$ satisfies the condition (ii). Let M_1 be a normal maximal subgroup of M. Since $LM_1 \in \mathfrak{L}$ and $L = C_G(L)$, we have $M_1 = 1$. This implies that $|M|$ is prime. Hence $M \in \mathfrak{L}$ since $\pi(\mathfrak{L}) = \mathbb{P}$. But then $G \in \mathfrak{F} = \mathfrak{N}\mathfrak{L}$. Therefore G is not soluble.

Let $q \neq p$ be any prime divisor of $|G|$. Suppose that G is not q-nilpotent. Then G has a q-closed $\mathfrak{N}$-critical subgroup $H = Q \rtimes R$ by [7, IV, Theorem 5.4], where Q is a Sylow q-subgroup of H, R is a cyclic Sylow r-subgroup of H. Since G is not soluble, $H \neq G$. Hence $H \leqslant M \in F(p) = \mathfrak{G}_p\mathfrak{L}$ for some maximal subgroup M of G. Since $M \in \mathfrak{G}_p\mathfrak{L}$, $M^{\mathfrak{L}} \leqslant O_p(M)$ and hence $H^{\mathfrak{L}} \leqslant Q \cap O_p(H) = 1$. This shows that H is nilpotent. This contradiction shows that G is q-nilpotent for all primes $q \neq p$. This induces that $G^{\mathfrak{N}}$ is a p-subgroup of G and thereby G is soluble. This contradiction completes the proof of (ii). $\square$

Remark 4.10. The condition "$\mathfrak{L}$ is a hereditary saturated formation satisfying the boundary condition in the class of all soluble groups" cannot be omitted in the statement (i). Indeed, let $\mathfrak{F} = \mathfrak{N}\mathfrak{U}$ and $G = P \rtimes A_4$, where P is a simple $\mathbb{F}_3A_4$-module P which is faithful for A_4. Let F be the canonical local satellite of $\mathfrak{F}$. Then $F(2) = \mathfrak{G}_2\mathfrak{U}$ by Lemma 2.9(2). Therefore G is an $F(2)$-critical group and $G \notin \mathfrak{F}$. Thus G does not satisfy the boundary condition in the class of all soluble groups.

Corollary 4.11. *Let $\mathfrak{F}$ be the class of all groups with $G' \leqslant F(G)$. Then $\mathfrak{F}$ satisfies the boundary condition.*

From Corollary 4.11 and Theorem A, we obtain

Corollary 4.12. *Let D be the intersection of all maximal subgroups H of G with the property $H' \leqslant F(H)$. Then D is the largest normal subgroup of G such that $D' \leqslant F(D)$ and G induces an abelian group of automorphisms on every chief factor of G below D.*

Note that Corollary 4.12 can be also found in [3, Corollary 7 and Remark 4].

Following [4, Chapter VII, Definition 6.9] we write $l(G)$ to denote the nilpotent length of the group G. Recall that $\mathfrak{N}^r$ is the product of r copies of $\mathfrak{N}$; $\mathfrak{N}^0$ is the class of groups of order 1 by definition. It is well known that $\mathfrak{N}^r$ is the class of all soluble groups G with $l(G) \leqslant r$. It is also known that $\mathfrak{N}^r$ is a hereditary saturated formation (see, for example, [4, p. 358]). Hence from Proposition 4.9 we get

Corollary 4.13. *Let $\mathfrak{F} = \mathfrak{N}^r\mathfrak{L}$ ($r \in \mathbb{N}$), where $\mathfrak{L}$ is a subformation of the formation of all abelian groups with $\pi(\mathfrak{L}) = \mathbb{P}$. Then $\mathfrak{F}$ satisfies the boundary condition in the class of all soluble groups.*

From Proposition 4.9 and Theorem A we get

Corollary 4.14. *Let D be the intersection of all maximal metanilpotent subgroups of G. Then D is the largest normal subgroup of G such that D is metanilpotent and G induces a nilpotent group of automorphisms on every chief factor of G below D.*

It is clear that every subnormal subgroup is a K-$\mathfrak{F}$-subnormal subgroup as well. On the other hand, in the case when $\mathfrak{N} \subseteq \mathfrak{F}$, every K-$\mathfrak{F}$-subnormal subgroup of any soluble subgroup G is $\mathfrak{F}$-subnormal in G. Hence from Theorem D and the above corollaries we get

Corollary 4.15. *Let $\mathfrak{F}$ be the class of all groups G with $G' \leqslant F(G)$. Then*:

(i) *The subgroup $Z_{\mathfrak{F}}(G)$ may be characterized as the intersection of all non-subnormal $\mathfrak{F}$-maximal subgroups of G, for each group G.*

(ii) *The subgroup $Z_{\mathfrak{F}}(G)$ may be characterized as the intersection of all non-$\mathfrak{F}$-subnormal $\mathfrak{F}$-maximal subgroups of G, for each soluble group G.*

Corollary 4.16. *Let $\mathfrak{F}$ be one of the following formations*:

(1) *the class of all nilpotent groups* (*Baer* [1]);

(2) *the class of all groups G with $G' \leqslant F(G)$* (*Skiba* [13]);

(3) *the class of all p-decomposable groups* (*Skiba* [12]).

Then for each group G, the equality $\mathrm{Int}_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G)$ holds.

Corollary 4.17. *(Sidorov [9].) Let $\mathfrak{F}$ be the class of all soluble groups G with $l(G) \leqslant r$ ($r \in \mathbb{N}$). Then for each soluble group G, the equality $Z_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ holds.*

Let $p_1 > p_2 > \cdots > p_r$ be the distinct primes dividing $|G|$, P_i a Sylow p_i-subgroup of G. Then G is said to satisfy *the Sylow tower property* if all subgroups $P_1, P_1P_2, \ldots, P_1P_2 \ldots P_{r-1}$ are normal in G. It is well known that every supersoluble group satisfies the Sylow tower property. As an application of Theorem C(a) we have the following result.

Proposition 4.18. *Let $\mathfrak{F}$ be the formation of all supersoluble groups and $\mathfrak{M}$ be the formation of all groups satisfying the Sylow tower property. Then $\mathrm{Int}_{\mathfrak{F}}(G) \leqslant \mathrm{Int}_{\mathfrak{M}}(G)$ for any group G.*

Proof. Let F be the canonical local satellite of $\mathcal{F}$. Then $F(p) = \mathcal{G}_p\mathcal{A}(p-1)$ for primes p (see Remark 4.8). Let G be any $F(p)$-critical group satisfying the Sylow tower property. We show that $G \in \mathcal{F}$. Let q be the largest prime dividing $|G|$, P the Sylow q-subgroup of G. If $G = P$, then clearly $G \in \mathcal{F}$. Let $P \neq G$. Then every Sylow subgroup of G belongs to $F(p)$. Hence $q = p$ and if E is a Hall p'-subgroup of G, then $E \in \mathcal{A}(p-1)$. But then $G \in F(p)$, a contradiction. Therefore $G \in \mathcal{F}$. This shows that $\mathcal{F}$ satisfies the boundary condition in $\mathcal{M}$. Hence, in view of Theorem C(a), we have $\mathrm{Int}_{\mathcal{F}}(G) \leqslant \mathrm{Int}_{\mathcal{M}}(G)$. $\square$

Remark 4.19. If $\mathcal{F} \subseteq \mathcal{M}$ are hereditary saturated formations, then for every group G we have $Z_{\mathcal{F}}(G) \leqslant Z_{\mathcal{M}}(G)$ and $Z_{\mathcal{F}}(G) \leqslant \mathrm{Int}_{\mathcal{F}}(G)$ by Lemma 2.4(g). Therefore, if $\mathcal{F}$ satisfies the boundary condition, then $\mathrm{Int}_{\mathcal{F}}(G) \leqslant \mathrm{Int}_{\mathcal{M}}(G)$ for every group G. But, by Remark 4.8, $\mathcal{F}$ does not necessarily satisfy the boundary condition. Hence we cannot deduce Proposition 4.18 from Theorem A.

Lemma 4.20. *Let $\mathcal{F} = LF(F)$ be a saturated formation, where F is the canonical local satellites of $\mathcal{F}$. Let $\mathcal{M} = \mathcal{F} \cap \mathcal{S}$ and M be the canonical local satellites of $\mathcal{M}$. Then $M(p) = F(p) \cap \mathcal{S}$ for all primes p.*

Proof. It is clear that $\mathcal{M} = LF(F_1)$, where $F_1(p) = F(p) \cap \mathcal{S}$ for all primes p. On the other hand, for any prime p we have $F_1(p) \subseteq \mathcal{M}$, and $F_1(p) = \mathcal{G}_p F_1(p)$ since

$$\mathcal{G}_p F_1(p) = \mathcal{G}_p\bigl(F(p) \cap \mathcal{S}\bigr) \subseteq F(p) \cap \mathcal{S}.$$

Hence $F_1 = M$ is the canonical local satellite of $\mathcal{M}$. $\square$

Lemma 4.21. *Let $(1) \neq \mathcal{F} = LF(F)$ be a saturated formation and $\mathcal{M} = \mathcal{F} \cap \mathcal{S}$. Let $\pi \subseteq \pi(\mathcal{F})$. Then $\mathcal{F}$ satisfies the boundary condition in the class $\mathcal{S}$ of all soluble groups if and only if $\mathcal{M}$ satisfies the boundary condition in $\mathcal{S}$.*

Proof. Let F and M be the canonical local satellites of the formations $\mathcal{F}$ and $\mathcal{M}$, respectively. Let $p \in \pi$. Then $M(p) = F(p) \cap \mathcal{S}$ by Lemma 4.20. Hence a soluble group G is $F(p)$-critical group if and only if G is $M(p)$-critical. On the other hand, the soluble group G belongs to $\mathcal{F}$ if and only if G belongs to $\mathcal{M}$. $\square$

In view of Lemmas 4.20 and 4.21, we may similarly prove the following general form of Theorem B.

Theorem 4.22. *Let $\mathcal{F}$ be a saturated formation such that $\mathcal{M} = \mathcal{F} \cap \mathcal{S}$ is hereditary and $(1) \neq \mathcal{M} \neq \mathcal{S}$. Let $\pi \subseteq \pi(\mathcal{F})$. Then the equality*

$$Z_{\pi\mathcal{F}}(G) = \mathrm{Int}_{\mathcal{F}}(G)$$

holds for each soluble group G if and only if $\mathcal{N} \subseteq \mathcal{F} = \mathcal{S}_{\pi'}\mathcal{F}$ and $\mathcal{F}$ satisfies the π-boundary condition in the class of all soluble groups.

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